Gravitational scattering of quantum fields and spacetime geometry

Takashi Uno and Gerardo Muñoz
Department of Physics, California State University, Fresno
Fresno, CA 93740-0037

ABSTRACT

We compute the lowest order quantum correction to the elastic scattering cross section for a massive spin-0 particle interacting with an external gravitational field. The result shows that quantum effects induce a breakdown of the geometrical picture afforded by the classical principle of equivalence.

PACS: 04.20.Cv, 11.80.-m, 04.62.+v, 11.10.Ef

I. INTRODUCTION

The issue of quantum corrections to general relativity has attracted the attention of physicists for many decades now. In the absence of a consistent theory of quantum gravity, one is limited to computing properties in the low-energy, long-distance regime, where one may convincingly argue that the results are reliable. A recent example of this is the computation of the leading quantum correction to the Newtonian potential by Donoghue [1]. A deeper question is whether these low-energy calculations shed some light on the physical reasons behind the failure of the program to quantize general relativity. It is the purpose of this paper to show by means of a rather simple example that the answer to this question is in the affirmative. Our work may be considered a proof that an observation made by Greenberger [2] within the context of Newtonian gravity and quantum mechanics remains valid in the domain of relativistic quantum fields in a post-Newtonian gravitational background.

A few words are necessary in order to clearly define the problem we wish to consider. Using the formalism of perturbation theory in the interaction picture of quantum theory, we shall calculate the differential cross section for a massive spin-0 field scattered by a weak external gravitational field (the scalar field was specifically chosen so that spin effects would not obscure the interpretation of our results). By external we mean a gravitational field that is - in contrast to the spin-0 field - purely classical and hence completely specified by general relativity. Given that we are interested only in the first quantum corrections, this has the advantage of avoiding the difficulties associated with graviton loops, a welcome simplification that does not alter our basic conclusions as far as the equivalence principle is concerned (more on this in Sec. IV). Furthermore, we shall concentrate on the *elastic* cross section; gravitational Bremsstrahlung with its attendant infrared divergence will not be included in our analysis.

It was shown by Golowich, Gribosky and Pal [3] that the lowest order terms agree with the general relativistic expressions. We shall therefore focus on the quantum effects, and simply refer the reader to their work for additional details on the classical cross section. We use natural units, $\hbar = c = 1$, and a metric with signature $(-,+,+,+)$, as in Ref. [3].

II. THE S-MATRIX IN THE INTERACTION PICTURE

In this section we give a brief overview of the method we shall follow. This is convenient to define our notation, and to remind the reader that the present situation differs from the typical gauge theory scenario in that the interaction term involves derivatives of the fields (see Sec. III below). Useful references are Sakurai [4], Scadron [5], and Itzykson and Zuber [6].

We specify the dynamics of the quantum field ϕ by means of the Lagrangian density $L = L_o + L_{\text{int}}$. From it, we compute the Hamiltonian density $H = \pi\,\dot{\phi} - L$, where π is the momentum conjugate to ϕ, $\pi \equiv \partial L / \partial \dot{\phi}$, and $\dot{\phi} \equiv \partial \phi / \partial t$.

Breaking H up into a free part plus an interaction part, dropping the free part, and interpreting the operators in the interaction part as free-field operators we obtain the Hamiltonian density H_{int} in the interaction picture. The S-matrix is constructed as a perturbation series [6]:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots \int d^4x_n \;\; T\left\{ H_{\text{int}}(x_1) \cdots H_{\text{int}}(x_n) \right\} \tag{1}$$

where T denotes the time-ordered product (A and B are arbitrary boson operators and θ is the unit step function):

$$T\left\{ A(x_1)\, B(x_2) \right\} = \theta(t_1 - t_2)\, A(x_1)\, B(x_2) + \theta(t_2 - t_1)\, B(x_2)\, A(x_1) \tag{2}$$

If the initial state $|i\rangle$ corresponds to a particle of momentum $\boldsymbol{p}$ and energy $\omega_{\boldsymbol{p}}$, and the final state $|f\rangle$ to a particle of momentum $\boldsymbol{p}'$ and energy $\omega_{\boldsymbol{p}'}$, we may anticipate conservation of energy and write

$$S_{fi} = \delta_{fi} - \frac{2\pi i}{\sqrt{4\,\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}V^2}}\; \delta(\omega_{\boldsymbol{p}'} - \omega_{\boldsymbol{p}})\, M_{fi} \tag{3}$$

The amplitude M_{fi} leads us to our final goal, the differential scattering cross section:

$$d\sigma = \frac{V\omega_{\boldsymbol{p}}}{|\boldsymbol{p}|}\, \frac{V}{(2\pi)^3} \int d^3\boldsymbol{p}'\; \frac{2\pi}{4\,\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}V^2}\; \delta(\omega_{\boldsymbol{p}'} - \omega_{\boldsymbol{p}})\left| M_{fi}\right|^2 \tag{4}$$

Here $|\boldsymbol{p}|/(V\omega_{\boldsymbol{p}})$ is the incident flux.

In the following section we shall compute the S-matrix by means of Eq. (1), extract the amplitude M_{fi} and use Eq. (4) to find the elastic cross section.

III. GRAVITATIONAL SCATTERING TO SECOND ORDER IN THE COUPLING

The Lagrangian density for a massive spin-0 field interacting with gravity is:

$$L = -\tfrac{1}{2}\sqrt{-g}\left(\partial_\mu \phi\, g^{\mu\nu}\, \partial_\nu \phi + m^2 \phi^2 \right) \tag{5}$$

We have included the square root of the determinant of the metric, $\sqrt{-g}$, in the Lagrangian density because our point of view is to regard gravity as a weak perturbation of flat spacetime. In particular, integrations will always be over d^4x.

In isotropic coordinates, the nonvanishing components of the spacetime metric due to a point mass M are given by (see, e.g., Stephani [7])

$$g_{00} = -\left(\frac{1 - GM/2r}{1 + GM/2r}\right)^2 \tag{6}$$

$$g_{ij} = \left(1 + GM/2r\right)^4 \delta_{ij} \tag{7}$$

Hence

$$\sqrt{-g} = \left(1 - GM/2r\right)\left(1 + GM/2r\right)^5 \tag{8}$$

and a straightforward computation using the definition of the conjugate momentum

$$\pi \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\left(1 + GM/2r\right)^7}{1 - GM/2r}\,\dot{\phi} \tag{9}$$

yields the Hamiltonian density

$$H = \frac{1}{2}\frac{1 - GM/2r}{\left(1 + GM/2r\right)^7}\,\pi^2 + \frac{1}{2}\left(1 - G^2M^2/4r^2\right)(\nabla\phi)^2$$
$$+ \frac{1}{2}\left(1 - G^2M^2/4r^2\right)\left(1 + GM/2r\right)^4 m^2\phi^2 \tag{10}$$

For weak fields we may expand Eq. (10) in powers of GM/r. The interaction Hamiltonian is then obtained after subtraction of the free Hamiltonian

$$H_{flat} = \frac{1}{2}\left\{\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right\} \tag{11}$$

Performing the appropriate transformation to the interaction picture we find, to second order in GM/r:

$$H_{\text{int}} = H_{\text{int}}^{(1)} + H_{\text{int}}^{(2)} \tag{12}$$

where

$$H_{\text{int}}^{(1)} = -2\,\frac{GM}{r}\,(N\dot{\phi}^2 - \tfrac{1}{2}m^2\,N\phi^2) \tag{13}$$

$$H_{\text{int}}^{(2)} = \frac{1}{8}\left(\frac{GM}{r}\right)^2\left\{35\,N\dot{\phi}^2 - N(\nabla\phi)^2 + 5\,m^2 N\phi^2\right\} \tag{14}$$

In Eqs. (13) and (14) N denotes the usual normal ordering of the operators that follow. Note that $H_{\text{int}} \neq -L_{\text{int}}$: gravitational scattering of a scalar field differs in this respect from the usual situation in quantum field theory (e.g., QED).

We are now ready to compute the S-matrix using Eq. (1):

$$S \approx 1 - i\int d^4x\, H_{\text{int}}^{(1)}(x) - i\int d^4x\, H_{\text{int}}^{(2)}(x)$$

$$+ \frac{(-i)^2}{2!}\int d^4x_1\int d^4x_2\; T\left\{H_{\text{int}}^{(1)}(x_1)\,H_{\text{int}}^{(1)}(x_2)\right\} \tag{15}$$

The calculation of $S_{fi} = \langle f \,|\, S \,|\, i \rangle$ is most conveniently carried out by expanding the scalar field in momentum eigenstates

$$\phi(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\,\omega_{\boldsymbol{k}} V}} \left\{ a_{\boldsymbol{k}}\, e^{ik\cdot x} + a_{\boldsymbol{k}}^{\dagger}\, e^{-ik\cdot x} \right\} \tag{16}$$

where $a_{\boldsymbol{k}}^{\dagger}$ and $a_{\boldsymbol{k}}$ are the creation and annihilation operators, respectively, and, with our choice of metric, $k \cdot x = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t$.

It is a relatively straightforward exercise to show that

$$\int d^4x \left\langle f \,\middle|\, H^{(1)}_{\text{int}}(x) \,\middle|\, i \right\rangle = -16\pi^2 GM \,\frac{\omega_{\boldsymbol{p}}^2 - \frac{1}{2} m^2}{V\omega_{\boldsymbol{p}}}\, \frac{\delta(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}'})}{(\boldsymbol{p} - \boldsymbol{p}')^2} \tag{17}$$

and

$$\int d^4x \left\langle f \middle| H^{(2)}_{\text{int}}(x) \middle| i \right\rangle = \pi^3 (GM)^2 \,\frac{35\omega_{\boldsymbol{p}}^2 - \boldsymbol{p}\cdot\boldsymbol{p}' + 5m^2}{2V\omega_{\boldsymbol{p}}}\, \frac{\delta(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}'})}{|\,\boldsymbol{p} - \boldsymbol{p}'\,|} \tag{18}$$

Evaluation of the last term in Eq. (15) requires more effort. We begin with the time-ordered product $T\left\{ H^{(1)}_{\text{int}}(x_1)\, H^{(1)}_{\text{int}}(x_2) \right\} \equiv T_{12}$:

$$T_{12} = \frac{4G^2M^2}{r_1 r_2}\; T\left\{ (N\dot{\phi}_1^2)(N\dot{\phi}_2^2) - \tfrac{1}{2} m^2 \left[(N\dot{\phi}_1^2)(N\phi_2^2) \right.\right.$$
$$\left.\left. + (N\phi_1^2)(N\dot{\phi}_2^2) \right] + \tfrac{1}{4} m^4 (N\phi_1^2)(N\phi_2^2) \right\} \tag{19}$$

The formulas below are needed in the calculation of $\langle f \,|\, T_{12} \,|\, i \rangle$. With the notation $\phi_i \equiv \phi(x_i)$, $\dot{\phi}_i \equiv \partial_{0i}\, \phi(x_i) \equiv \partial\, \phi(x_i) / \partial t_i$,

$$\frac{1}{2}\,\Delta_{F\,ij} \equiv \langle 0|\, T(\phi_i\ \phi_j)\,|0\rangle = -\frac{i}{(2\pi)^4}\int d^4k\ \frac{e^{ik\cdot(x_i - x_j)}}{k^2 + m^2 - i\,\varepsilon} \tag{20}$$

and

$$\frac{1}{2}\,\ddot{\Delta}_{F\,ij} \equiv \langle 0|\, T(\dot{\phi}_i\ \dot{\phi}_j)\,|0\rangle \tag{21}$$

one finds from Wick's theorem:

$$T\Big\{(N\phi_i^2)(N\phi_j^2)\Big\} = N(\phi_i^2\ \phi_j^2) + 2\,\Delta_{F\,ij}\,N(\phi_i\ \phi_j) + \frac{1}{2}\,\Delta_{F\,ij}^2 \tag{22}$$

$$T\Big\{(N\dot{\phi}_i^2)(N\phi_j^2)\Big\} = N(\dot{\phi}_i^2\ \phi_j^2) + 2(\partial_{0i}\,\Delta_{F\,ij})\,N(\dot{\phi}_i\,\phi_j) + \frac{1}{2}(\partial_{0i}\,\Delta_{F\,ij})^2 \tag{23}$$

$$T\Big\{(N\dot{\phi}_i^2)(N\dot{\phi}_j^2)\Big\} = N(\dot{\phi}_i^2\ \dot{\phi}_j^2) + 2\,\ddot{\Delta}_{F\,ij}\,N(\dot{\phi}_i\ \dot{\phi}_j) + \frac{1}{2}\,\ddot{\Delta}_{F\,ij}^2 \tag{24}$$

It is important to realize that $\ddot{\Delta}_{F\,ij} \neq \partial_{0i}\,\partial_{0j}\,\Delta_{F\,ij}$. Indeed, the definition of the time-ordered product, Eq. (2), may be recast in the form

$$T(\phi_i\ \phi_j) = \phi_i\ \phi_j - \theta(t_j - t_i)[\phi_i\,,\phi_j] \tag{25}$$

Using Eq. (25) in addition to the known properties of the distribution $i\,\Delta_{ij} \equiv [\phi_i\,,\phi_j]$,

$$\partial_{0i}\,\Delta_{ij}\Big|_{t_i = t_j} = -\,\partial_{0j}\,\Delta_{ij}\Big|_{t_i = t_j} = -\,\delta^3(\boldsymbol{x}_i - \boldsymbol{x}_j) \tag{26}$$

$$\Delta_{ij}\Big|_{t_i = t_j} = 0 \tag{27}$$

we obtain

$$\partial_{\mu i}\,\partial_{\nu j}\,T(\phi_i\ \phi_j) \;=\; T(\partial_{\mu i}\,\phi_i\ \partial_{\nu j}\,\phi_j) + i\,\eta_{\mu 0}\ \eta_{\nu 0}\ \delta^4(x_i - x_j) \tag{28}$$

In particular,

$$\ddot{\Delta}_{F\,ij} = \partial_{0i}\,\partial_{0j}\,\Delta_{F\,ij} - 2i\,\delta^4(x_i - x_j) \tag{29}$$

We are now ready to return to Eq. (19). Since $|i\rangle$ and $|f\rangle$ are one-particle states, we have

$$\langle f\,|\,N(\phi_i^2\ \phi_j^2)|i\rangle = \langle f\,|\,N(\dot{\phi}_i^2\ \phi_j^2)|i\rangle = \langle f\,|\,N(\phi_i^2\ \dot{\phi}_j^2)|i\rangle$$

$$= \langle f\,|\,N(\dot{\phi}_i^2\ \dot{\phi}_j^2)|i\rangle = 0 \tag{30}$$

The terms containing $\Delta_{F\,ij}^2$, $(\partial_{0i}\,\Delta_{F\,ij})^2$, and $\ddot{\Delta}_{F\,ij}^2$ will not contribute either if we demand that the final momentum $\boldsymbol{p}'$ be different from the initial momentum $\boldsymbol{p}$, since then $\langle f\,|\,i\rangle = 0$. Hence, after substitution of Eqs. (22) - (24), Eq. (19) implies:

$$\langle f\,|\,T_{12}\,|i\rangle \;=\; \frac{8G^2M^2}{r_1\,r_2}\,\Big\{\ddot{\Delta}_{F\,12}\,\langle f\,|\,N(\dot{\phi}_1\ \dot{\phi}_2)|i\rangle$$

$$-\tfrac{1}{2}\,m^2\Big[(\partial_{01}\,\Delta_{F\,12})\langle f\,|\,N(\dot{\phi}_1\ \phi_2)|i\rangle + (\partial_{02}\,\Delta_{F\,12})\langle f\,|\,N(\phi_1\ \dot{\phi}_2)|i\rangle\Big]$$

$$+\tfrac{1}{4}\,m^4\ \Delta_{F\,12}\,\langle f\,|\,N(\phi_1\ \phi_2)|i\rangle\Big\} \tag{31}$$

The matrix elements of the remaining normal-ordered products are easily evaluated:

$$\langle f\,|\,N(\phi_1\ \phi_2)|i\rangle \;=\; \frac{1}{2V\sqrt{\omega_{\boldsymbol{p}}\,\omega_{\boldsymbol{p}'}}}\Big\{\,e^{i(p\cdot x_2 - p'\cdot x_1)} \;+\; e^{i(p\cdot x_1 - p'\cdot x_2)}\Big\} \tag{32}$$

$$\langle f\,|N(\dot{\phi}_1\;\phi_2)|i\,\rangle \;=\; \frac{i}{2V}\left\{\sqrt{\frac{\omega_{\boldsymbol{p}'}}{\omega_{\boldsymbol{p}}}}\;e^{i(p\cdot x_2 - p'\cdot x_1)} \;-\; \sqrt{\frac{\omega_{\boldsymbol{p}}}{\omega_{\boldsymbol{p}'}}}\;e^{i(p\cdot x_1 - p'\cdot x_2)}\right\} \tag{33}$$

$$\langle f\,|N(\phi_1\;\dot{\phi}_2)|i\,\rangle \;=\; \frac{i}{2V}\left\{\sqrt{\frac{\omega_{\boldsymbol{p}'}}{\omega_{\boldsymbol{p}}}}\;e^{i(p\cdot x_1 - p'\cdot x_2)} \;-\; \sqrt{\frac{\omega_{\boldsymbol{p}}}{\omega_{\boldsymbol{p}'}}}\;e^{i(p\cdot x_2 - p'\cdot x_1)}\right\} \tag{34}$$

$$\langle f\,|\,N(\dot{\phi}_1\;\dot{\phi}_2)|i\,\rangle \;=\; \frac{\sqrt{\omega_{\boldsymbol{p}}\,\omega_{\boldsymbol{p}'}}}{2V}\left\{\,e^{i(p\cdot x_2 - p'\cdot x_1)} \;+\; e^{i(p\cdot x_1 - p'\cdot x_2)}\right\} \tag{35}$$

Integrating Eq. (31) over x_1 and x_2 is a straightforward task with the help of Eqs. (20), (29) and (32) - (35), if we regularize the infrared divergences by regarding the Newtonian potential GM/r as the limit $\mu \to 0$ of the Yukawa potential $GM\,e^{-\mu r}/r$. The result is:

$$\int d^4x_1 \int d^4x_2\,\langle f\,|\,T_{12}\,|i\,\rangle \;=\; -\frac{32\,i\,(GM)^2}{V\,\omega_{\boldsymbol{p}}}\,\delta(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}'})$$

$$\times\left\{(\omega_{\boldsymbol{p}}^2 - \tfrac{1}{2}m^2)^2\,K \;+\; \frac{2\,\pi^3\,\omega_{\boldsymbol{p}}^2}{|\,\boldsymbol{p} - \boldsymbol{p}'\,|}\right\} \tag{36}$$

where

$$K \;=\; 2\int \frac{d^3k}{(\boldsymbol{k}^2 - \boldsymbol{p}^2 - i\,\varepsilon)[(\boldsymbol{k} - \boldsymbol{p})^2 + \mu^2][(\boldsymbol{k} - \boldsymbol{p}')^2 + \mu^2]} \tag{37}$$

It is shown in the Appendix that, in the limit $\mu \to 0$, Eq. (37) yields

$$\lim_{\mu\to 0} K \;=\; -\frac{\pi^2\,i}{|\boldsymbol{p}|^3\,\sin^2(\theta/2)}\;\ln\frac{\mu}{2|\boldsymbol{p}|\sin(\theta/2)} \tag{38}$$

with θ the scattering angle, $\boldsymbol{p}\cdot\boldsymbol{p}' = \boldsymbol{p}^2\cos\theta$. Eq. (38) clearly shows the logarithmic nature of the infrared divergence.

Substituting Eqs. (17), (18) and (36) into Eq. (15) and comparing with Eq. (3), we get the following expression for the amplitude M_{fi}:

$$M_{fi} = -4\pi GM \frac{\omega_{\boldsymbol{p}}^2 - \frac{1}{2}m^2}{|\boldsymbol{p}|^2\sin^2(\theta/2)}$$

$$\times\left\{1 + \frac{\pi}{16}GM\frac{29\omega_{\boldsymbol{p}}^2 + |\boldsymbol{p}|^2\cos\theta - 5m^2}{\omega_{\boldsymbol{p}}^2 - \frac{1}{2}m^2}|\boldsymbol{p}|\sin(\theta/2)\right.$$

$$\left. - 4iGM\frac{\omega_{\boldsymbol{p}}^2 - \frac{1}{2}m^2}{|\boldsymbol{p}|}\ln\frac{\mu}{2|\boldsymbol{p}|\sin(\theta/2)}\right\} \qquad (39)$$

The differential scattering cross section may be found from Eq. (4). To order $(GM)^3$:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{cl}\left\{1 + \frac{\pi}{8}GM\frac{29\omega_{\boldsymbol{p}}^2 + |\boldsymbol{p}|^2\cos\theta - 5m^2}{\omega_{\boldsymbol{p}}^2 - \frac{1}{2}m^2}|\boldsymbol{p}|\sin(\theta/2)\right\} \qquad (40)$$

$(d\sigma/d\Omega)_{cl}$ is the classical cross section [3],

$$\left(\frac{d\sigma}{d\Omega}\right)_{cl} = (GM)^2\frac{(\omega_{\boldsymbol{p}}^2 - \frac{1}{2}m^2)^2}{|\boldsymbol{p}|^4\sin^4(\theta/2)} \qquad (41)$$

Eq. (40) is the central result of this paper. Notice that the infrared-divergent term does not enter the cross section: to the accuracy of our calculation, it amounts to only a phase in the amplitude M_{fi}.

IV. CONCLUSIONS

Despite the simplifying assumptions made, result (40) for the scattering cross section exposes a remarkable consequence of the dynamics of quantum fields in a gravitational background. If one writes Eq. (41) in terms of kinematical quantities, the classical cross section turns out to be independent of the mass m of the spin-0 field, as expected on the basis of the equivalence principle. The correction term in Eq. (40), however, is proportional to this mass. This implies that the equivalence principle (more precisely, Galileo's principle of equivalence in the terminology used by Ohanian and Ruffini [8]) ceases to hold in quantum theory. Since the cross section is no longer independent of the mass of the particle, the geometrical point of view adopted in general relativity loses its appeal. A similar conclusion was reached by Greenberger [2] within the context of nonrelativistic quantum mechanics (see also the recent paper by Sonego [9]). Experimental confirmation of a mass dependence in interference experiments in gravitational fields was first provided by Colella, Overhauser and Werner [10].

We should point out that the equality of inertial and gravitational mass (Newton's principle of equivalence [8]) is not challenged by any of the above. Quite to the contrary, the experiment reported in Ref. [10] may be interpreted as supporting the idea that $m_I = m_G$, and we have incorporated this equality into our work as a fundamental assumption. Thus two possible formulations of the physical principle behind general relativity, which in the classical theory may - for all practical purposes - be treated as basically equivalent, end up being drastically different when applied to the quantum domain.

At the risk of belaboring the obvious, it is perhaps worth reminding the reader that the only way in which quantum theory has entered our problem is through the scalar field ϕ. In particular, the gravitational field has been treated as classical. Aside from the ensuing technical simplification, this has the advantage of completely avoiding any reference to a quantum theory of gravity: the conclusions apply to a *quantum* field in a *classical*, general-relativistic background (contrary to Ref. [1], which deals with an external spinless source and quantum gravity effects.)

When does the idea of a spacetime geometry become a valid concept? An answer to this question may readily be provided by inserting the appropriate factors of $\hbar$ and c back into Eq. (40). The correction term is thus seen to be of order $GMp/c^2\hbar \propto R_S/\lambda$, where R_S is the Schwarzschild radius of the source and λ the wavelength of the scattered particle. Our perturbative calculation is justified provided $R_S/\lambda << 1$, i.e., in the regime where the particle can probe regions which are large compared to the characteristic volume over which the gravitational field varies appreciably. Of course, a violation of the equivalence principle by extended objects is a trivial proposition even in classical physics. It is therefore important to realize that a mass-independent behavior arises in Eq. (40) in the limit as $\lambda \to \infty$: the notion of a spacetime geometry emerges in this particular instance as an effective property in the nontrivial limit of long wavelengths.

Our computations have specifically excluded radiative corrections to the cross section. While these will certainly change the mathematical form of Eq. (40), they will not change the qualitative conclusions regarding the principle of equivalence we have derived from it. The reason is simply that these radiative effects cannot be used to cancel the quantum correction in Eq. (40) so as to restore the mass independence of the cross section. Indeed, given a term with a fixed power of the mass M generating the external gravitational field (a fixed number of external lines in the language of Feynman diagrams), radiative corrections will contribute loops involving virtual gravitons and spin-0 particles. But these loops carry with them specific powers of the mass m, and so can never be expected to cancel a term with a different number of loops. Thus radiative corrections are irrelevant here.

APPENDIX

The integral (37) may be computed using the Feynman parametrization [11]:

$$\frac{1}{a_1 a_2 \cdots a_n} = \Gamma(n) \int_0^1 dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-2}} dz_{n-1}$$

$$\times \frac{1}{[a_1(1-z_1) + a_2(z_1 - z_2) + \cdots + a_n z_{n-1}]^n} \qquad \text{(A1)}$$

A proof of Eq. (A1) as well as a generalization of this formula may be found in almost any book on advanced quantum mechanics or field theory (see, e. g., Ref. [12]).

With the help of Eq. (A1) we may rewrite Eq. (37) as

$$K = 4 \int_0^1 dz_1 \int_0^{z_1} dz_2 \int \frac{d^3 s}{[\boldsymbol{s}^2 - \gamma - i\,\varepsilon(1 - z_1)]^3} \qquad \text{(A2)}$$

where we have used the new variables

$$\boldsymbol{s} = \boldsymbol{k} - [(z_1 - z_2)\,\boldsymbol{p} + z_2\,\boldsymbol{p}'] \qquad \text{(A3)}$$

$$\gamma = \boldsymbol{p}^2\,[(1 - z_1)^2 - 4 z_2 (z_1 - z_2) \sin^2(\theta/2)] - \mu^2 z_1 \qquad \text{(A4)}$$

The advantage of the Feynman parametrization is now obvious: the integral over $\boldsymbol{s}$ is much simpler than the previous integral over $\boldsymbol{k}$. Using contour methods we obtain

$$\int \frac{d^3 s}{[\boldsymbol{s}^2 - \gamma - i\,\varepsilon\,(1 - z_1)]^3} \;=\; -\,\frac{\pi^2\, i}{4\,\gamma^{3/2}} \tag{A5}$$

The remaining integrals over z_1 and z_2 may be found in tables. Taking the limit as $\varepsilon \to 0$, and denoting $\bar{\mu} \equiv \mu / |\boldsymbol{p}|$:

$$K = -\frac{i\,\pi^2}{|\boldsymbol{p}|^3} \int_0^1 dz_1 \int_0^{z_1} dz_2 \; \frac{1}{[(1 - z_1)^2 - 4\,z_2\,(z_1 - z_2)\sin^2(\theta/2)] - \bar{\mu}^2 z_1\,]^{3/2}}$$

$$= -\,\frac{i\,\pi^2}{|\boldsymbol{p}|^3} \int_0^1 \frac{z_1\, dz_1}{[(1 - z_1)^2 - z_1^2 \sin^2(\theta/2) \,-\, \bar{\mu}^2 z_1]\,\sqrt{(1 - z_1)^2 - \bar{\mu}^2 z_1}} \tag{A6}$$

Finally:

$$K \;=\; \frac{\pi^2}{|\boldsymbol{p}|^3\,\rho\,\sin(\theta/2)} \left\{ 2\,\tan^{-1}\left[\frac{\bar{\mu}\,\sin(\theta/2)}{\rho}\right] \right.$$

$$\left. +\; i\,\ln\left[\frac{\rho \,+\, 2\sin(\theta/2)}{\rho \,-\, 2\sin(\theta/2)}\right] \right\} \tag{A7}$$

with $\rho = \sqrt{\bar{\mu}^2(\bar{\mu}^2 + 4) \,+\, 4\sin^2(\theta/2)}$. In the limit $\bar{\mu} \to 0$, Eq. (A7) reduces to Eq. (38).

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